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PURELY TRANSVERSE WAVES IN ELASTIC ANISOTROPIC MEDIA

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Formulas are obtained for decompositions of the third- and fourth-rank tensors symmetric in the last two and three indices, respectively, into irreducible parts invariant relative to the orthogonal group of coordinate transformation. The corresponding parts of the decompositions are orthogonal to each other. These decompositions are used to obtain a general representation of the displacement vectors of plane transverse waves in elastic isotropic and anisotropic solids. It is shown that the displacement vectors of transverse waves are second-, third-, and fourth-degree homogeneous polynomials of the wave normal. Special orthotropic materials are found that transmit purely transverse waves for any direction of the wave normal. The eigenmoduli, eigenstates, and engineering constants (bulk moduli, Young's moduli, Poisson's ratios, shear moduli, and Lamé constants of the closest isotropic materials) are determined for these materials.

Key words: *irreducible invariant decomposition, longitudinal and transverse waves, anisotropy, elastic moduli, eigenmoduli, eigenstate.*

This paper develops the approaches proposed in [1, 2]. Finding purely transverse waves and anisotropic materials that transmit such waves is of fundamental importance in crystal physics and geophysics [3, 4].

Ignoring body forces, we write the equations of elasticity for arbitrary anisotropy in Cartesian coordinates x_1, x_2, and x_3:

$$L_{ij} u_j = 0, \qquad L_{ij} = L_{ji} = A_{i(kl)j}\,\partial_{kl} - \rho\delta_{ij}\,\partial_{44}. \tag{1}$$

Here u_j is the displacement vector, $A_{i(kl)j} = (A_{iklj} + A_{ilkj})/2$, $A_{iklj} = A_{kilj} = A_{ljik}$ is the elastic-modulus tensor, ρ is the constant density of the material, ∂_k is the derivative with respect to the coordinate x_k, ∂_4 is the derivative with respect to time $x_4 = t$, and δ_{ij} is the Kronecker symbol. The summation is performed over repeated letter indices, and the indices in parentheses denote symmetrization.

For an isotropic material, the operator (1) is written as

$$L_{ij} = (\lambda + \mu)\partial_{ij} + \delta_{ij}\,(\mu\,\partial_{kk} - \rho\,\partial_{44}), \tag{2}$$

where λ and μ are the Lamé constants. If for operators (1) and (2) there exist differential operators $T = [t_{jp}]$, $D = \operatorname{diag}(D_1, D_2, D_3)$ and $D_i = a^{(i)}_{kl}\,\partial_{kl} - \rho\,\partial_{44}$ with constant coefficients such that

$$LT = TD, \qquad |T| \neq 0, \tag{3}$$

the general solution of Eqs. (1) is given by [5, 6]

$$u = T\varphi, \qquad D\varphi = f, \qquad Tf = 0. \tag{4}$$

The formulas $u = T\varphi$, $\varphi = T'\tilde{u}$, and $L\tilde{u} = 0$ transform the solutions of the equations $Lu = 0$ and $D\varphi = 0$ into one another. The prime denotes the transpose of the matrix. The expression $u = TT'\tilde{u}$ is the formula producing new solutions, i.e., $Q = TT'$ is a symmetry operator [6, 7].

Relation (3) implies that t_{jp} $(p = 1, 2, 3)$ are eigenvectors and D_i are eigenvalues (operators) for L. Replacing ∂_k by n_k (wave-normal vector) and ∂_{44} by $v^2 = v_i v_i$ $(v_i = vn_i)$ and setting $D_i = 0$, we reduce relation (3) to the

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Christoffel equation, from which the displacement vectors t_{jp} and squared phase velocities of plane waves v^2 are determined [3].

We write (3) as

$$(L_{ij} - \delta_{ij} D_1) T_{j1} = 0, \qquad (L_{ij} - \delta_{ij} D_2) T_{j2} = 0, \qquad (L_{ij} - \delta_{ij} D_3) T_{j3} = 0. \tag{5}$$

Let $D_1 = a_{kl}\,\partial_{kl} - \rho\,\partial_{44}$, $a_{kl} = a_{(kl)}$ and

$$T_{j1} = \gamma_j + \gamma_{js}\,\partial_s + \gamma_{j(pq)}\,\partial_{pq} + \gamma_{j(pqr)}\,\partial_{pqr} + \dots\ . \tag{6}$$

Similar expressions can be written for D_2, D_3, T_{j2}, and T_{j3}. From (5) and (6), we obtain

$$(L_{ij} - \delta_{ij} D_1) T_{j1} = (A_{i(kl)j} - \delta_{ij} a_{kl}) \partial_{kl} T_{j1}$$

$$= (A_{i(kl)j} - \delta_{ij} a_{kl}) \gamma_j\,\partial_{kl} + (A_{i(kl)j} - \delta_{ij} a_{kl}) \gamma_{js}\,\partial_{kls}$$

$$+ (A_{i(kl)j} - \delta_{ij} a_{kl}) \gamma_{j(pq)}\,\partial_{klpq} + (A_{i(kl)j} - \delta_{ij} a_{kl}) \gamma_{j(pqr)}\,\partial_{klpqr} + \dots\ . \tag{7}$$

Relations (5) hold if the symmetrized coefficients of $\partial_{kl}, \partial_{kls}, \partial_{klpq}, \partial_{klpqr}, \dots$ in (7) vanish:

$$(A_{i(kl)j} - \delta_{ij} a_{kl}) \gamma_j = 0;$$

$$(1/3)[(A_{i(kl)j} - \delta_{ij} a_{kl}) \gamma_{js} + (A_{i(ks)j} - \delta_{ij} a_{ks}) \gamma_{jl} + (A_{i(ls)j} - \delta_{ij} a_{ls}) \gamma_{jk}] = 0; \tag{8}$$

$$(1/6)[(A_{i(kl)j} - \delta_{ij} a_{kl}) \gamma_{j(pq)} + (A_{i(kp)j} - \delta_{ij} a_{kp}) \gamma_{j(lq)} + (A_{i(kq)j} - \delta_{ij} a_{kq}) \gamma_{j(lp)}$$

$$+ (A_{i(lp)j} - \delta_{ij} a_{lp}) \gamma_{j(kq)} + (A_{i(lq)j} - \delta_{ij} a_{lq}) \gamma_{j(kp)} + (A_{i(pq)j} - \delta_{ij} a_{pq}) \gamma_{j(kl)}] = 0; \tag{9}$$

$$(1/10)[(A_{i(kl)j} - \delta_{ij} a_{kl}) \gamma_{j(pqr)} + (A_{i(kp)j} - \delta_{ij} a_{kp}) \gamma_{j(lqr)} + (A_{i(kq)j} - \delta_{ij} a_{kq}) \gamma_{j(lpr)}$$

$$+ (A_{i(kr)j} - \delta_{ij} a_{kr}) \gamma_{j(lpq)} + (A_{i(lp)j} - \delta_{ij} a_{lp}) \gamma_{j(kqr)} + (A_{i(lq)j} - \delta_{ij} a_{lq}) \gamma_{j(kpr)}$$

$$+ (A_{i(lr)j} - \delta_{ij} a_{lr}) \gamma_{j(kpq)} + (A_{i(pq)j} - \delta_{ij} a_{pq}) \gamma_{j(klr)} + (A_{i(pr)j} - \delta_{ij} a_{pr}) \gamma_{j(klq)} + (A_{i(qr)j} - \delta_{ij} a_{qr}) \gamma_{j(klp)}] = 0, \ . \tag{10}$$

Setting the free indices in (8)–(10) equal to 1, 2, and 3, we obtain the corresponding system of equations for unknowns $A_{i(kl)j} - \delta_{ij} a_{kl}$, γ_{js}, $\gamma_{j(pq)}$, $\gamma_{j(pqr)}, \dots$. Systems of the form (8), (9) were considered in [6, 8].

It is obvious that for the operator (2), $t_{j1} = \partial_j$ is an eigenvector and determines purely longitudinal wave [3] for any direction of the wave normal. In [4–6, 9], anisotropic materials were found that transmit purely longitudinal waves for any direction of the wave normal. Given the propagation direction of transverse waves, one can completely solve the Christoffel equation [3]. In [5, 6, 9], purely transverse waves were obtained:

$$t_{j2} = \varepsilon_{jms} c_m\,\partial_s, \qquad t_{j3} = c_j\,\partial_{kk} - c_m\,\partial_{mj}. \tag{11}$$

Here ε_{jms} is a Levi-Civita antisymmetric tensor and c_j is a nonzero vector. For t_{j2} in (11), we obtain the coefficients $\gamma_{js} = \varepsilon_{jms} c_m$, and the elastic-constant tensor A_{iklj} of a transversely isotropic material with the rotation axis c_j satisfies Eqs. (8). If $c_j = (0, 0, 1)$, then $t_{j2} = (-\partial_2, \partial_1, 0)$ is a purely transverse wave that for any direction of the wave normal n_k (∂_k) can travel in a transversely isotropic material [8] with the rotation axis x_3; the phase velocity in this case is $\rho v^2 = (A_{11} - A_{21})(n_1^2 + n_2^2)/2 + A_{44} n_3^2/2$. Here A_{ij} is the elastic-modulus matrix that corresponds to the tensor A_{ijkl}. Obviously, the purely transverse waves (11) are also eigenvectors for the operator (2) in the case of an isotropic material [8].

We find all transverse waves of the form $t_{i2} = a_{ijk}\,\partial_{jk} = a_{i(jk)}\,\partial_{jk}$ or $t_{i2} = a_{ijkl}\,\partial_{jkl} = a_{i(jkl)}\,\partial_{jkl}$, where $a_{ijk} = a_{i(jk)}$ and $a_{ijkl} = a_{i(jkl)}$ are tensors symmetric in the last two and three indices, respectively. These vectors are orthogonal to $t_{i1} = \partial_i$, i.e., the equalities $t_{i1} t_{i2} = a_{ijk}\,\partial_{ijk} = a_{(ijk)}\,\partial_{ijk} = 0$, and $t_{i1} t_{i2} = a_{ijkl}\,\partial_{ijkl} = a_{(ijkl)}\,\partial_{ijkl} = 0$, should hold, from which it follows that $a_{(ijk)} = 0$ and $a_{(ijkl)} = 0$.

In a similar way as was done in [10], the tensors a_{ijk} and a_{ijkl} can be decomposed into invariant parts that correspond to the irreducible linear representations of the orthogonal group of coordinate transformations:

$$a_{ijk} = a_{i(jk)} = c^{(1)}_{ijk} + c^{(2)}_{ijk} + (\varepsilon_{ijl} H_{lk} + \varepsilon_{ikl} H_{lj})/2 + S_{ijk},$$

$$c^{(1)}_{ijk} = a_1 g_i \delta_{jk} + a_2 (g_j \delta_{ki} + g_k \delta_{ji})/2, \qquad c^{(2)}_{ijk} = b_1 h_i \delta_{jk} + b_2 (h_j \delta_{ki} + h_k \delta_{ji})/2; \tag{12}$$

$$a_{ijkl} = a_{i(jkl)} = a\delta_{i(j}\delta_{kl)} + D_{ijkl} + N_{ijkl} + g_m\varepsilon_{mi(j}\delta_{kl)} + d_{ijkl} + \varepsilon_{mi(j}S_{kl)m}$$

$$= a(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk})/3 + D_{ijkl} + N_{ijkl} + g_m(\varepsilon_{mij}\delta_{kl} + \varepsilon_{mik}\delta_{lj} + \varepsilon_{mil}\delta_{jk})/3 + d_{ijkl}$$

$$+ (\varepsilon_{mij}S_{klm} + \varepsilon_{mik}S_{ljm} + \varepsilon_{mil}S_{jkm})/3, \tag{13}$$

$$D_{ijkl} = \alpha_1(H_{ij}\delta_{kl} + H_{ik}\delta_{lj} + H_{il}\delta_{jk}) + \alpha_2(H_{jk}\delta_{li} + H_{kl}\delta_{ji} + H_{lj}\delta_{ki})/3,$$

$$d_{ijkl} = \beta_1(h_{ij}\delta_{kl} + h_{ik}\delta_{lj} + h_{il}\delta_{jk}) + \beta_2(h_{jk}\delta_{li} + h_{kl}\delta_{ji} + h_{lj}\delta_{ki})/3.$$

Here (a_1, a_2), (b_1, b_2) and (α_1, α_2), (β_1, β_2) are independent pairs of arbitrary real numbers, a is a constant, g_i and h_i are vectors, $H_{ij} = H_{(ij)}$ and $h_{ij} = h_{(ij)}$ are deviators: $H_{ii} = 0$ and $h_{ii} = 0$, $S_{ijk} = S_{(ijk)}$ is a septor (a symmetric traceless tensor of rank three), and $N_{ijkl} = N_{(ijkl)}$ is a nonor (symmetric traceless tensor of rank four). The free parameters a_i, b_i, α_i, and β_i can be chosen such that all parts in (12) and (13) are orthogonal to one another. All quantities on the right sides of (12) and (13) can be uniquely expressed in terms of the tensors $a_{i(jk)}$ and $a_{i(jkl)}$. Conversely, these tensors can be specified by formulas (12) and (13).

The tensors $c^{(1)}_{ijk}$ and $c^{(2)}_{ijk}$ in (12) are normalized and orthogonal if [2]

$$a_1 = \frac{1}{\sqrt{3}}\Big(\omega_{11} - \frac{1}{\sqrt{5}}\,\omega_{21}\Big), \quad a_2 = \sqrt{\frac{3}{5}}\,\omega_{21}; \quad b_1 = \frac{1}{\sqrt{3}}\Big(\omega_{12} - \frac{1}{\sqrt{5}}\,\omega_{22}\Big), \quad b_2 = \sqrt{\frac{3}{5}}\,\omega_{22}.$$

Here ω_{ij} is an arbitrary orthogonal 2×2 matrix of the second order: $\omega_{ip}\omega_{iq} = \delta_{pq}$. The vectors, deviator, and septor on the right side of (12) are uniquely determined by the tensor a_{ijk} [2]:

$$g_i = \frac{(b_1 + 2b_2)a_{ikk} - (3b_1 + b_2)a_{ssi}}{5(a_1b_2 - a_2b_1)},$$

$$h_i = \frac{(3a_1 + a_2)a_{ssi} - (a_1 + 2a_2)a_{ikk}}{5(a_1b_2 - a_2b_1)}, \qquad a_1b_2 - a_2b_1 \neq 0; \tag{14}$$

$$H_{lk} = (a_{ijk}\varepsilon_{ijl} + a_{ijl}\varepsilon_{ijk})/3 = 2a_{ij(k}\varepsilon_{l)ij}/3, \quad S_{ijk} = a_{(ijk)} - (a_1 + a_2)g_{(i}\delta_{jk)} - (b_1 + b_2)h_{(i}\delta_{jk)}.$$

If the vector parts in (12) are orthogonal, instead of (14) we obtain

$$g_i = \frac{a_1a_{ikk} + a_2a_{ssi}}{(\sqrt{3}a_1 + a_2/\sqrt{3})^2 + 5a_2^2/3}, \qquad h_i = \frac{b_1a_{ikk} + b_2a_{ssi}}{(\sqrt{3}b_1 + b_2/\sqrt{3})^2 + 5b_2^2/3}.$$

Setting $a_1 = 1/3$, $a_2 = 2/3$, $b_1 = 2/3$, and $b_2 = -2/3$, from (12), we obtain the decomposition [11]

$$a_{ijk} = (g_i\delta_{jk} + g_j\delta_{ki} + g_k\delta_{ji})/3 + S_{ijk} + (2h_i\delta_{jk} - h_j\delta_{ki} - h_k\delta_{ji})/3 + (\varepsilon_{ijl}H_{lk} + \varepsilon_{ikl}H_{lj})/2 \tag{15}$$

into symmetric and nonsymmetric parts.

For transverse waves, the symmetric part in (15) is $a_{(ijk)} = 0$, i.e., the tensor a_{ijk} becomes

$$a_{ijk} = (2h_i\delta_{jk} - h_j\delta_{ki} - h_k\delta_{ji})/3 + (\varepsilon_{ijl}H_{lk} + \varepsilon_{ikl}H_{lj})/2.$$

From this, we obtain the transverse wave (eigenvector)

$$t_{j2} = a_{jsk}\,\partial_{sk} = 2(h_j\partial_{kk} - h_k\,\partial_{kj})/3 + \varepsilon_{jsl}H_{lk}\,\partial_{sk} = c_j\,\partial_{kk} - c_k\,\partial_{kj} + \varepsilon_{jsl}H_{lk}\,\partial_{sk},$$

$$c_j = 2h_j/3 \tag{16}$$

and the third eigenvector

$$t_{j3} = \varepsilon_{jmn}\,\partial_m t_{n2} = \varepsilon_{jmn}c_n\,\partial_{mss} + H_{lp}\,\partial_{lpj} - H_{jp}\,\partial_{pss}.$$

One can easily verify that the matrix

$$T = [\partial_j,\ c_j\,\partial_{kk} - c_m\,\partial_{mj} + \varepsilon_{jsl}H_{lp}\,\partial_{sp},\ \varepsilon_{jmn}c_n\,\partial_{mss} + H_{lp}\,\partial_{lpj} - H_{jp}\,\partial_{pss}] \tag{17}$$

and the operator (2) satisfy relation (3) and $D_1 = (\lambda + 2\mu)\,\partial_{kk} - \rho\,\partial_{44}$ and $D_2 = D_3 = \mu\,\partial_{kk} - \rho\,\partial_{44}$. For a nonzero deviator H_{lk}, the displacement vectors of the transverse waves t_{j2} and t_{j3} are second- and third-degree homogeneous polynomials of the wave normal n_k (∂_k). Taking into account (17) and using formulas (4), we obtain a new representation [2] of the solution of the Lamé equation for an isotropic material.

We now consider the tensor $a_{ijkl} = a_{i(jkl)}$. A decomposition of the form (13) can be obtained using the transformation

$$a^*_{ijkl} = a_{(jkl)i} = (a_{jkli} + a_{klji} + a_{ljki})/3.$$

We find the projectors

$$p_{ijkl} = \alpha a_{ijkl} + \beta(a_{jkli} + a_{klji} + a_{ljki})/3. \tag{18}$$

Since the double action of the projector yields a projector, the coefficients in (18) satisfy the equations

$$3\alpha^2 + \beta^2 = 3\alpha, \qquad 6\alpha\beta + 2\beta^2 = 3\beta,$$

whose solutions are (1, 0), (1/4, 3/4), and (3/4, −3/4). In this case, we obtain the following projectors: $p^{(2)}_{ijkl} = (a_{ijkl}+a_{jkli}+a_{klji}+a_{ljki})/4 = a_{(ijkl)}$ (symmetrization over all indices) and $p^{(3)}_{ijkl} = (3a_{ijkl}-a_{jkli}-a_{klji}-a_{ljki})/4 = a_{ijkl} - a_{(ijkl)}$ (nonsymmetric part). These projectors are orthogonal $p^{(2)}_{ijkl}p^{(3)}_{ijkl} = 0$ and their sum is equal to the identical projector $p^{(1)}_{ijkl} = a_{ijkl}$.

From (13), we find the convolution of the tensors

$$D_{ijkl}d_{ijkl} = [15\alpha_1\beta_1 + 2(\alpha_1\beta_2 + \alpha_2\beta_1) + 5\alpha_2\beta_2/3]H_{ij}h_{ij} \tag{19}$$

and the squared norms of the tensors

$$\begin{aligned} D_{ijkl}D_{ijkl} &= (15\alpha_1^2 + 4\alpha_1\alpha_2 + 5\alpha_2^2/3)H_{ij}H_{ij}, \\ d_{ijkl}d_{ijkl} &= (15\beta_1^2 + 4\beta_1\beta_2 + 5\beta_2^2/3)h_{ij}h_{ij}. \end{aligned} \tag{20}$$

The tensors in (19) and (20) are orthogonal and normalized provided that

$$\begin{aligned} 15\alpha_1^2 + 4\alpha_1\alpha_2 + 5\alpha_2^2/3 &= (\sqrt{15}\,\alpha_1 + 2\alpha_2/\sqrt{15}\,)^2 + (\sqrt{7/5}\,\alpha_2)^2 = 1, \\ 15\beta_1^2 + 4\beta_1\beta_2 + 5\beta_2^2/3 &= (\sqrt{15}\beta_1 + 2\beta_2/\sqrt{15})^2 + (\sqrt{7/5}\,\beta_2)^2 = 1, \\ 15\alpha_1\beta_1 + 2(\alpha_1\beta_2 + \alpha_2\beta_1) + 5\alpha_2\beta_2/3 &= (\sqrt{15}\alpha_1 + 2\alpha_2/\sqrt{15})(\sqrt{15}\beta_1 + 2\beta_2/\sqrt{15}) + \sqrt{7/5}\,\alpha_2\sqrt{7/5}\,\beta_2 = 0. \end{aligned} \tag{21}$$

Relations (21) imply that

$$\begin{bmatrix} \sqrt{15}\,\alpha_1 + 2\alpha_2/\sqrt{15} & \sqrt{15}\,\beta_1 + 2\beta_2/\sqrt{15} \\ \sqrt{7/5}\,\alpha_2 & \sqrt{7/5}\,\beta_2 \end{bmatrix} = \omega_{ij}$$

is an arbitrary orthogonal 2×2 matrix: $\omega_{ip}\omega_{iq} = \delta_{pq}$. Moreover,

$$\begin{aligned} \alpha_1 &= \frac{1}{\sqrt{5}}\Big(\frac{1}{\sqrt{3}}\,\omega_{11} - \frac{2}{3\sqrt{7}}\,\omega_{21}\Big), \qquad \alpha_2 = \sqrt{\frac{5}{7}}\,\omega_{21}; \\ \beta_1 &= \frac{1}{\sqrt{5}}\Big(\frac{1}{\sqrt{3}}\,\omega_{12} - \frac{2}{3\sqrt{7}}\,\omega_{22}\Big), \qquad \beta_2 = \sqrt{\frac{5}{7}}\,\omega_{22} \end{aligned} \tag{22}$$

and D_{ijkl} and d_{ijkl} are normalized and orthogonal tensors.

The constant, vector, deviators, septor, and nonor on the right side of (13) are uniquely determined by the tensor $a_{ijkl} = a_{i(jkl)}$:

$$\begin{aligned} &a = a_{iikk}/5, \qquad g_s = 3\varepsilon_{sij}a_{ijkk}/10; \qquad H_{ij} = \frac{(5\beta_1 + 2\beta_2/3)p_{ij} - (2\beta_1 + 5\beta_2/3)q_{ij}}{7(\alpha_2\beta_1 - \alpha_1\beta_2)}, \\ &h_{ij} = \frac{-(5\alpha_1 + 2\alpha_2/3)p_{ij} + (2\alpha_1 + 5\alpha_2/3)q_{ij}}{7(\alpha_2\beta_1 - \alpha_1\beta_2)}, \\ &\alpha_2\beta_1 - \alpha_1\beta_2 \neq 0, \quad p_{ij} = a_{ssij} - a_{sskk}\delta_{ij}/3, \quad q_{ij} = (a_{ijkk} + a_{jikk})/2 - a_{sskk}\delta_{ij}/3; \end{aligned} \tag{23}$$

$$S_{kls} = 3(a_{ij(kl}\varepsilon_{s)ij} - \delta_{(kl}\varepsilon_{s)ij}a_{ijpp}/5)/4 = [\varepsilon_{sij}a_{ijkl} + \varepsilon_{kij}a_{ijls} + \varepsilon_{lij}a_{ijsk} - (\varepsilon_{sij}\delta_{kl} + \varepsilon_{kij}\delta_{ls} + \varepsilon_{lij}\delta_{sk})a_{ijpp}/5]/4,$$

$$N_{ijkl} = a_{(ijkl)} - a\delta_{i(j}\delta_{kl)} - (3\alpha_1 + \alpha_2)H_{(ij}\delta_{kl)} - (3\beta_1 + \beta_2)h_{(ij}\delta_{kl)}.$$

For the parameters (22), the deviators (23) become

$$H_{ij} = \omega[-\sqrt{5/7}\,\omega_{12}p_{ij} + (2\omega_{12}/\sqrt{7} + \sqrt{3}\,\omega_{22})q_{ij}/\sqrt{5}],$$

$$h_{ij} = \omega[\sqrt{5/7}\,\omega_{11}p_{ij} - (2\omega_{11}/\sqrt{7} + \sqrt{3}\,\omega_{21})q_{ij}/\sqrt{5}], \quad \omega = |\omega_{ij}| = \omega_{11}\omega_{22} - \omega_{21}\omega_{12} = \pm 1.$$

For $\alpha_1 = 1/6$, $\alpha_2 = 1/2$, $\beta_1 = 1/6$, and $\beta_2 = -1/2$, from (13) we obtain the decomposition [11]

$$a_{ijkl} = a_{i(jkl)} = a(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk})/3 + N_{ijkl}$$

$$+ (H_{ij}\delta_{kl} + H_{ik}\delta_{lj} + H_{il}\delta_{jk} + H_{jk}\delta_{li} + H_{kl}\delta_{ji} + H_{lj}\delta_{ki})/6 + g_m(\varepsilon_{mij}\delta_{kl} + \varepsilon_{mik}\delta_{lj} + \varepsilon_{mil}\delta_{jk})/3$$

$$+ (h_{ij}\delta_{kl} + h_{ik}\delta_{lj} + h_{il}\delta_{jk} - h_{jk}\delta_{li} - h_{kl}\delta_{ji} - h_{lj}\delta_{ki})/6 + (\varepsilon_{mij}S_{klm} + \varepsilon_{mik}S_{ljm} + \varepsilon_{mil}S_{jkm})/3, \tag{24}$$

whose parts are orthogonal to one another.

For the transverse waves, the symmetric part in (24) is $a_{(ijkl)} = 0$, i.e., the tensor a_{ijkl} has the form

$$a_{ijkl} = g_m(\varepsilon_{mij}\delta_{kl} + \varepsilon_{mik}\delta_{lj} + \varepsilon_{mil}\delta_{jk})/3$$

$$+ (h_{ij}\delta_{kl} + h_{ik}\delta_{lj} + h_{il}\delta_{jk} - h_{jk}\delta_{li} - h_{kl}\delta_{ji} - h_{lj}\delta_{ki})/6 + (\varepsilon_{mij}S_{klm} + \varepsilon_{mik}S_{ljm} + \varepsilon_{mil}S_{jkm})/3. \tag{25}$$

Taking into account (25), we obtain the transverse wave (eigenvector)

$$t_{i2} = a_{ijkl}\partial_{jkl} = g_m\varepsilon_{mij}\,\partial_{jkk} + (h_{ij}\,\partial_{jkk} - h_{jk}\,\partial_{ijk})/2 + \varepsilon_{mij}S_{klm}\,\partial_{jkl}$$

and third eigenvector

$$t_{j3} = \varepsilon_{jsn}\,\partial_s t_{n2} = g_m\,\partial_{mjkk} - g_j\,\partial_{ppkk} + \varepsilon_{jsl}h_{lp}\,\partial_{spkk}/2 + S_{klm}\,\partial_{klmj} - S_{jkl}\,\partial_{klpp}.$$

One can readily verify that the matrix

$$T = [\partial_j,\ \varepsilon_{mjn}g_m\,\partial_{nkk} + (h_{jp}\,\partial_{pss} - h_{lp}\,\partial_{lpj})/2 + \varepsilon_{jpm}S_{mkl}\,\partial_{pkl},$$

$$g_m\,\partial_{mjkk} - g_j\,\partial_{ppkk} + \varepsilon_{jsl}h_{lp}\,\partial_{spkk}/2 + S_{klm}\,\partial_{klmj} - S_{jkl}\,\partial_{klpp}] \tag{26}$$

and the operator (2) satisfy relation (3) for the same values of D_1 and $D_2 = D_3$. For a nonzero septor S_{mkl}, the displacement vectors of transverse waves are third- and fourth-degree homogeneous polynomials of the wave normal n_k (∂_k). Using formulas (4) and taking into account (26), one obtains one more representation of the solution of the Lamé equations (2) for isotropic materials.

In [12, 13], a classification of the matrices L and T satisfying relation (3) is given and it is argued that degree of the vectors t_{jp} with respect to n_k (∂_k) is not higher than the second. However, the vector t_{j3} in the matrix (17) is of the third degree if $H_{lp} \neq 0$ and the vectors t_{j2} and t_{j3} in (26) are of the third and fourth degrees, respectively, if $S_{mkl} \neq 0$. It follows that the classification given in [12, 13] is incomplete. For $H_{lp} = 0$, relation (17) yields (11), and relation (26) yields (17) for $S_{mkl} = 0$.

Let the coordinate system be the principal coordinate system for the deviator H_{lk} in (16), i.e., $H_{21} = H_{31} = H_{32} = 0$, $H_{11} = H_1$, $H_{22} = H_2$, $H_{33} = H_3$, and $H_1 + H_2 + H_3 = 0$. From (16), we obtain

$$t_{12} = c_1(\partial_{22} + \partial_{33}) + (H_3 - H_2)\,\partial_{23} - c_3\,\partial_{13} - c_2\,\partial_{12},$$

$$t_{22} = c_2(\partial_{11} + \partial_{33}) - c_3\,\partial_{23} + (H_1 - H_3)\,\partial_{13} - c_1\,\partial_{12}, \tag{27}$$

$$t_{32} = c_3(\partial_{11} + \partial_{22}) - c_2\,\partial_{23} - c_1\,\partial_{13} + (H_2 - H_1)\,\partial_{12}.$$

We consider the case $c_j = 0$. Relation (27) becomes

$$t_{j2} = (h_1\,\partial_{23},\ h_2\,\partial_{13},\ h_3\,\partial_{12}),$$

$$h_1 = H_3 - H_2, \qquad h_2 = H_1 - H_3, \qquad h_3 = H_2 - H_1, \qquad h_1 + h_2 + h_3 = 0. \tag{28}$$

For the transverse wave (28), the coefficients $\gamma_{j(pq)}$ are given by

$$\gamma_{j(11)} = 0, \qquad \gamma_{j(22)} = 0, \qquad \gamma_{j(33)} = 0,$$
$$\gamma_{j(23)} = (h_1/2,\, 0,\, 0), \qquad \gamma_{j(13)} = (0,\, h_2/2,\, 0), \qquad \gamma_{j(12)} = (0,\, 0,\, h_3/2). \tag{29}$$

In view of (29), system (9) reduces to the equations

$$(A^*_{11} - a)h_1 + A^*_{66}h_2 + A^*_{55}h_3 = 0,$$
$$A^*_{66}h_1 + (A^*_{22} - a)h_2 + A^*_{44}h_3 = 0, \qquad A^*_{55}h_1 + A^*_{44}h_2 + (A^*_{33} - a)h_3 = 0. \tag{30}$$

Here $a_{11} = a_{22} = a_{33} = a$, $a_{23} = a_{13} = a_{12} = 0$, and A^*_{ik} is the matrix corresponding to the tensor $A_{i(kl)j}$ [14].

From (30) it follows that a is an eigenvalue and h_j is an eigenvector of the symmetric matrix

$$A^*_{ij} = \begin{bmatrix} A^*_{11} & A^*_{66} & A^*_{55} \\ A^*_{66} & A^*_{22} & A^*_{44} \\ A^*_{55} & A^*_{44} & A^*_{33} \end{bmatrix}. \tag{31}$$

Therefore, the matrix (31) can be written in terms of eigenvalues and eigenvectors:

$$A^*_{ij} = a_1 h_{i1}h_{j1} + a_2 h_{i2}h_{j2} + a_3 h_{i3}h_{j3}. \tag{32}$$

Here h_{ip} is an orthogonal matrix [15]:

$$h_{ip} = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{-(1+c)}{\sqrt{3[1+(c-1)^2+c^2]}} & \dfrac{c-1}{\sqrt{1+(c-1)^2+c^2}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{2-c}{\sqrt{3[1+(c-1)^2+c^2]}} & \dfrac{-c}{\sqrt{1+(c-1)^2+c^2}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{2c-1}{\sqrt{3[1+(c-1)^2+c^2]}} & \dfrac{1}{\sqrt{1+(c-1)^2+c^2}} \end{bmatrix}. \tag{33}$$

Here c is an arbitrary real parameter, and, obviously, $h_{1p} + h_{2p} + h_{3p} = 0$ for $p = 2, 3$.

In this case, the total matrix A^*_{ik} [14] has the form

$$A^*_{ik} = \begin{bmatrix} A^*_{11} & & & & & \\ a & A^*_{22} & & & \text{sym} & \\ a & a & A^*_{33} & & & \\ 0 & 0 & 0 & A^*_{44} & & \\ 0 & 0 & 0 & 0 & A^*_{55} & \\ 0 & 0 & 0 & 0 & 0 & A^*_{66} \end{bmatrix}, \tag{34}$$

where the diagonal elements are given by formulas (31)–(33) and the quantity a can take values a_2 or a_3. Taking into account (34), we write the operator (1) as

$$L_{ij} = \begin{bmatrix} A^*_{11}\partial_{11} + a(\partial_{22} + \partial_{33}) - \rho\partial_{44} & A^*_{66}\partial_{12} & A^*_{55}\partial_{13} \\ A^*_{66}\partial_{21} & a\partial_{11} + A^*_{22}\partial_{22} + a\partial_{33} - \rho\partial_{44} & A^*_{44}\partial_{23} \\ A^*_{55}\partial_{31} & A^*_{44}\partial_{32} & a(\partial_{11} + \partial_{22}) + A^*_{33}\partial_{33} - \rho\partial_{44} \end{bmatrix}. \tag{35}$$

One can easily verify that if Eqs. (30) are satisfied, the vector (28) is an eigenvector of the operator (35) and $D_2 = a\partial_{kk} - \rho\partial_{44}$. It is obvious that for *any direction* of the wave normal n_k (∂_k), the vector (28) is the displacement vector of *purely transverse* wave and the phase velocity is given by $\rho v^2 = a_2 n_k n_k = a_2$ or $\rho v^2 = a_3 n_k n_k = a_3$, depending on which column in (33) — h_{i2} or h_{i3} — is chosen for h_i in (28).

Using (34), we find (see [14]) the elastic-modulus matrix A_{ij} [1] in Hooke's law

$$A_{ij} = \begin{bmatrix} A^*_{11} & & & & & \\ A^*_{66} - a & A^*_{22} & & & \text{sym} & \\ A^*_{55} - a & A^*_{44} - a & A^*_{33} & & & \\ 0 & 0 & 0 & 2a & & \\ 0 & 0 & 0 & 0 & 2a & \\ 0 & 0 & 0 & 0 & 0 & 2a \end{bmatrix}, \tag{36}$$

where $a = a_2$ or $a = a_3$. It should be noted that the operator (35) [hence, the material (36)] does not admit the purely longitudinal wave $t_{j1} = \partial_j$ for any direction of the wave normal. This is possible for an isotropic material with the operator (2) [see formulas (17) and (26)] and for the anisotropic material considered in [4–6, 9].

Since the quantity a in (36) can take two values $a = a_2$ or $a = a_3$, two types of materials correspond to matrix (36). Obviously, these are subclasses of orthotropic materials.

Using formulas (30)–(33), one can show that the vectors h_{ip} ($p = 1,\ 2,\ 3$) in (33) are also eigenvectors for the first quarter of the matrix (36). It follows that the materials (36) have eigenmoduli [15]

$$\lambda_1 = a_1 - 2a_2, \quad \lambda_2 = 2a_2, \quad \lambda_3 = a_2 + a_3, \quad \lambda_4 = \lambda_5 = \lambda_6 = 2a_2 \tag{37}$$

or

$$\lambda_1 = a_1 - 2a_3, \quad \lambda_2 = a_2 + a_3, \quad \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 2a_3 \tag{38}$$

and eigenstates [15]

$$t_{ip} = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{-(1+c)}{\sqrt{3[1+(c-1)^2+c^2]}} & \dfrac{c-1}{\sqrt{1+(c-1)^2+c^2}} & 0 & 0 & 0 \\ \dfrac{1}{\sqrt{3}} & \dfrac{2-c}{\sqrt{3[1+(c-1)^2+c^2]}} & \dfrac{-c}{\sqrt{1+(c-1)^2+c^2}} & 0 & 0 & 0 \\ \dfrac{1}{\sqrt{3}} & \dfrac{2c-1}{\sqrt{3[1+(c-1)^2+c^2]}} & \dfrac{1}{\sqrt{1+(c-1)^2+c^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{39}$$

From (39) it follows that among the eigenstates, there are a spherical tensor and five orthogonal deviators, three of which are pure-shear tensors, i.e., deviators with zero determinants. For some values of c, however, there may be four pure-shear tensors.

For the physically real materials (36), the eigenmoduli (37) and (38) should be positive. This implies the following necessary and sufficient conditions of positive definiteness of the matrix (36)

$$\begin{gathered} a_1 - 2a_2 > 0, \qquad a_2 + a_3 > 0, \qquad a_2 > 0; \\ a_1 - 2a_3 > 0, \qquad a_2 + a_3 > 0, \qquad a_3 > 0. \end{gathered} \tag{40}$$

By virtue of the multiplicity of the eigenmoduli (37) and (38), materials of the form (36) can be written as

$$\begin{gathered} A_{ij} = (\lambda_1 - \lambda_2)t_{i1}t_{j1} + (\lambda_3 - \lambda_2)t_{i3}t_{j3} + \lambda_2\delta_{ij}; \\ A_{ij} = (\lambda_1 - \lambda_3)t_{i1}t_{j1} + (\lambda_2 - \lambda_3)t_{i2}t_{j2} + \lambda_3\delta_{ij}. \end{gathered} \tag{41}$$

Here the eigenmoduli λ_1, λ_2, and λ_3 are given by formulas (37) and (38), and the eigenstates t_{i1}, t_{i2}, and t_{i3} by formulas (39). In (41), the moduli λ_1, λ_2, and λ_3 are not ordered, i.e., they are numbered according to the notation (37) and (38). Depending on the relations between the moduli λ_1, λ_2, and λ_3, materials of the form (41) can belong to the classes $\{1,1,4\}$, $\{1,4,1\}$, and $\{4,1,1\}$ [16].

Thus, the anisotropic materials (41), which transmit *purely transverse* waves (28) for *any direction* of the wave normal n_k, depend on four parameters: a_1, a_2, a_3, and c. The first three parameters satisfy inequalities (40), and the parameter c, which determines the eigenstates (39), can take arbitrary real values. If $a_2 = a_3$ in (37) and (38), materials of the form (36) and (41) become isotropic.

The compliance matrix a_{ij} inverse to A_{ij} (41) is given by

$$\begin{gathered} a_{ij} = \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\Big)t_{i1}t_{j1} + \Big(\frac{1}{\lambda_3} - \frac{1}{\lambda_2}\Big)t_{i3}t_{j3} + \frac{1}{\lambda_2}\,\delta_{ij}; \\ a_{ij} = \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\Big)t_{i1}t_{j1} + \Big(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\Big)t_{i2}t_{j2} + \frac{1}{\lambda_3}\,\delta_{ij}. \end{gathered} \tag{42}$$

Using (42), we find the engineering constants [17, 18] for these materials. The bulk modulus K is written as

$$\frac{1}{K} = a_{iikk} = t_{iipq}\,\frac{1}{\lambda_{pqrs}}\,t_{kkrs} = \frac{1}{\lambda_1}\,t_{ii11}^2 + \frac{1}{\lambda_2}\,t_{ii22}^2 + \frac{1}{\lambda_3}\,t_{ii33}^2 + \frac{1}{\lambda_4}\,2t_{ii23}^2 + \frac{1}{\lambda_5}\,2t_{ii13}^2 + \frac{1}{\lambda_6}\,2t_{ii12}^2. \tag{43}$$

Here a_{ijkl} is the compliance-coefficient tensor corresponding to the matrix a_{ij}, $t_{ij11}, \ldots, \sqrt{2}t_{ij12}$ are the tensors of the eigenstates that correspond to the columns $t_{i1}, \ldots, t_{i6}$ in (39), and λ_{pqrs} is the diagonal tensor of the eigenmoduli. Since $t_{ij11} = \delta_{ij}/\sqrt{3}$ and $t_{ii11} = \sqrt{3}$ and the remaining eigenstates are deviators, i.e., $t_{iipq} = 0$ and $pq \neq 11$, from (37), (38), and (43) we obtain $3K = \lambda_1 = a_1 - 2a_2$ or $3K = \lambda_1 = a_1 - 2a_3$.

Let n_i and m_i $(i = 1, 2, 3)$ be two orthogonal directions: $n_i n_i = 1$, $m_i m_i = 1$, and $n_i m_i = 0$. We introduce the notation

$$\tilde{n}_i = (n_1^2,\, n_2^2,\, n_3^2,\, \sqrt{2}\,n_2 n_3,\, \sqrt{2}\,n_1 n_3,\, \sqrt{2}\,n_1 n_2),$$

$$\tilde{m}_i = (m_1^2,\, m_2^2,\, m_3^2,\, \sqrt{2}\,m_2 m_3,\, \sqrt{2}\,m_1 m_3,\, \sqrt{2}\,m_1 m_2), \tag{44}$$

$$\widetilde{nm}_i = (n_1 m_1,\, n_2 m_2,\, n_3 m_3,\, \sqrt{2}(n_2 m_3 + n_3 m_2)/2,\, \sqrt{2}(n_1 m_3 + n_3 m_1)/2,\, \sqrt{2}(n_1 m_2 + n_2 m_1)/2),$$

i.e., (44) are vectors that correspond to the symmetric tensors $n_i n_j$, $m_i m_j$, and $n_{(i} m_{j)}$. Young's modulus E_n in the direction n_i is written as

$$\frac{1}{E_n} = n_i n_j a_{ijkl} n_k n_l = n_i n_j t_{ijpq}\,\frac{1}{\lambda_{pqrs}}\,t_{klrs} n_k n_l = \frac{1}{\lambda_1}\,(t_{ij11} n_i n_j)^2 + \frac{1}{\lambda_2}\,(t_{ij22} n_i n_j)^2 + \frac{1}{\lambda_3}\,(t_{ij33} n_i n_j)^2$$

$$+ \frac{1}{\lambda_4}\,2(t_{ij23} n_i n_j)^2 + \frac{1}{\lambda_5}\,2(t_{ij13} n_i n_j)^2 + \frac{1}{\lambda_6}\,2(t_{ij12} n_i n_j)^2$$

$$= \tilde{n}_i a_{ij} \tilde{n}_j = \frac{1}{\lambda_1}\,(t_{i1}\tilde{n}_i)^2 + \frac{1}{\lambda_2}\,(t_{i2}\tilde{n}_i)^2 + \frac{1}{\lambda_3}\,(t_{i3}\tilde{n}_i)^2 + \frac{1}{\lambda_4}\,(t_{i4}\tilde{n}_i)^2 + \frac{1}{\lambda_5}\,(t_{i5}\tilde{n}_i)^2 + \frac{1}{\lambda_6}\,(t_{i6}\tilde{n}_i)^2. \tag{45}$$

Taking into account (37)–(39), (42), and (44), from (45) we obtain

$$\frac{1}{E_n} = \frac{4a_2 - a_1}{2a_2(a_1 - 2a_2)}\,(t_{i1}\tilde{n}_i)^2 + \frac{a_2 - a_3}{2a_2(a_2 + a_3)}\,(t_{i3}\tilde{n}_i)^2 + \frac{1}{2a_2}$$

$$= \frac{a_1 - a_2}{3a_2(a_1 - 2a_2)} + \frac{a_2 - a_3}{2a_2(a_2 + a_3)}\,\frac{[(c-1)n_1^2 - cn_2^2 + n_3^2]^2}{1 + (c-1)^2 + c^2};$$

$$\frac{1}{E_n} = \frac{4a_3 - a_1}{2a_3(a_1 - 2a_3)}\,(t_{i1}\tilde{n}_i)^2 + \frac{a_3 - a_2}{2a_3(a_2 + a_3)}\,(t_{i2}\tilde{n}_i)^2 + \frac{1}{2a_3}$$

$$= \frac{a_1 - a_3}{3a_3(a_1 - 2a_3)} + \frac{a_3 - a_2}{2a_3(a_2 + a_3)}\,\frac{[-(1+c)n_1^2 + (2-c)n_2^2 + (2c-1)n_3^2]^2}{3[1 + (c-1)^2 + c^2]}.$$

For tension in the direction n_i, Poisson's ratio ν_{mn} in the direction m_i has the form

$$\frac{\nu_{mn}}{E_n} = m_i m_j a_{ijkl} n_k n_l = m_i m_j t_{ijpq}\,\frac{1}{\lambda_{pqrs}}\,t_{klrs} n_k n_l$$

$$= \frac{1}{\lambda_1}\,(t_{ij11} m_i m_j)(t_{kl11} n_k n_l) + \frac{1}{\lambda_2}\,(t_{ij22} m_i m_j)(t_{kl22} n_k n_l) + \frac{1}{\lambda_3}\,(t_{ij33} m_i m_j)(t_{kl33} n_k n_l)$$

$$+ \frac{1}{\lambda_4}\,2(t_{ij23} m_i m_j)(t_{kl23} n_k n_l) + \frac{1}{\lambda_5}\,2(t_{ij13} m_i m_j)(t_{kl13} n_k n_l) + \frac{1}{\lambda_6}\,2(t_{ij12} m_i m_j)(t_{kl12} n_k n_l)$$

$$= \tilde{m}_i a_{ij} \tilde{n}_j = \frac{1}{\lambda_1}\,(t_{i1}\tilde{m}_i)(t_{j1}\tilde{n}_j) + \frac{1}{\lambda_2}\,(t_{i2}\tilde{m}_i)(t_{j2}\tilde{n}_j) + \frac{1}{\lambda_3}\,(t_{i3}\tilde{m}_i)(t_{j3}\tilde{n}_j)$$

$$+ \frac{1}{\lambda_4}\,(t_{i4}\tilde{m}_i)(t_{j4}\tilde{n}_j) + \frac{1}{\lambda_5}\,(t_{i5}\tilde{m}_i)(t_{j5}\tilde{n}_j) + \frac{1}{\lambda_6}\,(t_{i6}\tilde{m}_i)(t_{j6}\tilde{n}_j). \tag{46}$$

With allowance for (37)–(39), (42), and (44), from (46) we obtain

$$\frac{\nu_{mn}}{E_n} = \frac{4a_2 - a_1}{2a_2(a_1 - 2a_2)}\,(t_{i1}\tilde{m}_i)(t_{j1}\tilde{n}_j) + \frac{a_2 - a_3}{2a_2(a_2 + a_3)}\,(t_{i3}\tilde{m}_i)(t_{j3}\tilde{n}_j)$$

$$= \frac{4a_2 - a_1}{6a_2(a_1 - 2a_2)} + \frac{a_2 - a_3}{2a_2(a_2 + a_3)}\,\frac{[(c-1)m_1^2 - cm_2^2 + m_3^2][(c-1)n_1^2 - cn_2^2 + n_3^2]}{1 + (c-1)^2 + c^2};$$

$$\frac{\nu_{mn}}{E_n} = \frac{4a_3 - a_1}{2a_3(a_1 - 2a_3)}\,(t_{i1}\tilde{m}_i)(t_{j1}\tilde{n}_j) + \frac{a_3 - a_2}{2a_3(a_2 + a_3)}\,(t_{i2}\tilde{m}_i)(t_{j2}\tilde{n}_j)$$

$$= \frac{4a_3 - a_1}{6a_3(a_1 - 2a_3)} + \frac{a_3 - a_2}{2a_3(a_2 + a_3)}\,\frac{[-(1+c)m_1^2 + (2-c)m_2^2 + (2c-1)m_3^2][-(1+c)n_1^2 + (2-c)n_2^2 + (2c-1)n_3^2]}{3[1 + (c-1)^2 + c^2]}.$$

The shear modulus μ_{nm} in the plane determined by the normals n_i and m_i is given by

$$\frac{1}{4\mu_{nm}} = n_i m_j a_{ijkl} n_k m_l = n_i m_j t_{ijpq}\,\frac{1}{\lambda_{pqrs}}\,t_{klrs} n_k m_l$$

$$= \frac{1}{\lambda_1}\,(t_{ij11} n_i m_j)^2 + \frac{1}{\lambda_2}\,(t_{ij22} n_i m_j)^2 + \frac{1}{\lambda_3}\,(t_{ij33} n_i m_j)^2$$

$$+ \frac{1}{\lambda_4}\,2(t_{ij23} n_i m_j)^2 + \frac{1}{\lambda_5}\,2(t_{ij13} n_i m_j)^2 + \frac{1}{\lambda_6}\,2(t_{ij12} n_i m_j)^2$$

$$= \widetilde{nm}_i a_{ij} \widetilde{nm}_j = \frac{1}{\lambda_1}\,(t_{i1}\widetilde{nm}_i)^2 + \frac{1}{\lambda_2}\,(t_{i2}\widetilde{nm}_i)^2 + \frac{1}{\lambda_3}\,(t_{i3}\widetilde{nm}_i)^2$$

$$+ \frac{1}{\lambda_4}\,(t_{i4}\widetilde{nm}_i)^2 + \frac{1}{\lambda_5}\,(t_{i5}\widetilde{nm}_i)^2 + \frac{1}{\lambda_6}\,(t_{i6}\widetilde{nm}_i)^2. \tag{47}$$

Taking into account (37)–(39), (42), and (44), from (47) we obtain

$$\frac{1}{4\mu_{nm}} = \frac{4a_2 - a_1}{2a_2(a_1 - 2a_2)}\,(t_{i1}\widetilde{nm}_i)^2 + \frac{a_2 - a_3}{2a_2(a_2 + a_3)}\,(t_{i3}\widetilde{nm}_i)^2 + \frac{1}{2a_2}\,(\widetilde{nm}_i)(\widetilde{nm}_i)$$

$$= \frac{1}{4a_2} + \frac{a_2 - a_3}{2a_2(a_2 + a_3)}\,\frac{[(c-1)n_1 m_1 - cn_2 m_2 + n_3 m_3]^2}{1 + (c-1)^2 + c^2};$$

$$\frac{1}{4\mu_{nm}} = \frac{4a_3 - a_1}{2a_3(a_1 - 2a_3)}\,(t_{i1}\widetilde{nm}_i)^2 + \frac{a_3 - a_2}{2a_3(a_2 + a_3)}\,(t_{i2}\widetilde{nm}_i)^2 + \frac{1}{2a_3}\,(\widetilde{nm}_i)(\widetilde{nm}_i)$$

$$= \frac{1}{4a_3} + \frac{a_3 - a_2}{2a_3(a_2 + a_3)}\,\frac{[-(1+c)n_1 m_1 + (2-c)n_2 m_2 + (2c-1)n_3 m_3]^2}{3[1 + (c-1)^2 + c^2]}.$$

Thus, all engineering constants of materials (36), (41), and (42) are determined in general form.

Matrices (36) and (41) can be decomposed into invariant irreducible parts using the formulas given in [10]. In particular, the Lamé constants of the isotropic materials the closest to (41) are given by

$$\lambda = (2A_{sskk} - A_{skks})/15 = (5\lambda_1 - 4\lambda_2 - \lambda_3)/15 = (5a_1 - 19a_2 - a_3)/15,$$

$$2\mu = (3A_{skks} - A_{sskk})/15 = (4\lambda_2 + \lambda_3)/5 = (9a_2 + a_3)/5;$$

$$\lambda = (5\lambda_1 - \lambda_2 - 4\lambda_3)/15 = (5a_1 - a_2 - 19a_3)/15, \qquad 2\mu = (\lambda_2 + 4\lambda_3)/5 = (a_2 + 9a_3)/5.$$

Equations (30) admit one more solution if only one eigenvector in representation (32), say, h_{i3}, satisfies the condition $h_{13}+h_{23}+h_{33}=0$. In this case, $a=a_3$ in (36) and in the second formula (41) the eigenmoduli $\lambda_1>0$, $\lambda_2>0$, and $\lambda_3=\lambda_4=\lambda_5=\lambda_6=2a_3>0$ are independent quantities, and the eigenstates t_{ip} have the following form [15]:

$$t_{ip}=\begin{bmatrix} \frac{1}{\sqrt{1+c_3^2+(1+c_1(c_3-1))^2}} & \frac{-[c_3+(1+c_1(c_3-1))c_1]}{\sqrt{[1+c_3^2+(1+c_1(c_3-1))^2][1+(c_1-1)^2+c_1^2]}} & \frac{c_1-1}{\sqrt{1+(c_1-1)^2+c_1^2}} & 0 & 0 & 0 \\ \frac{c_3}{\sqrt{1+c_3^2+(1+c_1(c_3-1))^2}} & \frac{1-(1+c_1(c_3-1))(c_1-1)}{\sqrt{[1+c_3^2+(1+c_1(c_3-1))^2][1+(c_1-1)^2+c_1^2]}} & \frac{-c_1}{\sqrt{1+(c_1-1)^2+c_1^2}} & 0 & 0 & 0 \\ \frac{1+c_1(c_3-1)}{\sqrt{1+c_3^2+(1+c_1(c_3-1))^2}} & \frac{c_3(c_1-1)+c_1}{\sqrt{[1+c_3^2+(1+c_1(c_3-1))^2][1+(c_1-1)^2+c_1^2]}} & \frac{1}{\sqrt{1+(c_1-1)^2+c_1^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{48}$$

Here c_1 and c_3 are arbitrary real parameters. It follows that, in this case, the anisotropic material (41) transmitting the *purely transverse* wave (28) for *any direction* of the wave normal n_k depends on five parameters: λ_1, λ_2, a_3, c_1, and c_3.

Using (36), (41), and (48), one can show that the matrix (31) has an eigenvector $h_{i3}=t_{i3}$ $(i=1,2,3)$ and an eigenvalue a_3. The eigenvectors h_{i1} and h_{i2} in (32) have the structure of the vectors t_{i1} and t_{i2} $(i=1,\ 2,\ 3)$ in (48) with a different parameter c_3. There is no need to calculate the values of a_1 and a_2 in (32). For $c_3=1$, the matrix (48) becomes the matrix (39).

Formulas (43) and (45)–(47) are also used to calculate the engineering constants in the case of (41) and (48). The Lamé constants of the closest isotropic material are as follows:

$$\lambda=[(\lambda_1-2a_3)(2t_{kk11}^2-1)+(\lambda_2-2a_3)(2t_{kk22}^2-1)]/15,$$

$$2\mu=[(\lambda_1-2a_3)(3-t_{kk11}^2)+(\lambda_2-2a_3)(3-t_{kk22}^2)]/15+2a_3.$$

In [4], the following anisotropic material transmitting purely transverse waves for any direction of the wave normal is given as an example:

$$A_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+H_{ij}\delta_{kl}+H_{kl}\delta_{ij}=A_{ijkl}^{is}+H_{ij}\delta_{kl}+H_{kl}\delta_{ij}. \tag{49}$$

Here A_{ijkl}^{is} is the isotropic part and $H_{ij}=H_{(ij)}$ is a deviator: $H_{ii}=0$. In the principal axes of the deviator H_{ij}, the tensor (49) corresponds to the elastic-modulus matrix A_{ij}

$$A_{ij}=\begin{bmatrix} \lambda+2\mu+2H_1 & & & & & \\ \lambda-H_3 & \lambda+2\mu+2H_2 & & & \text{sym} & \\ \lambda-H_2 & \lambda-H_1 & \lambda+2\mu+2H_3 & & & \\ 0 & 0 & 0 & 2\mu & & \\ 0 & 0 & 0 & 0 & 2\mu & \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix}, \tag{50}$$

$$H_1+H_2+H_3=0.$$

The material (50) is a particular case of materials of the form (36), (41), and (48). For (50), the eigenmoduli λ_p, eigenstates t_{ip}, and parameters c_1 and c_3 are given by

$$\lambda_1=2\mu+[3\lambda+\sqrt{3(3\lambda^2+4H_iH_i)}]/2=2\mu+\tilde{\lambda}_1,$$

$$\lambda_2=2\mu+[3\lambda-\sqrt{3(3\lambda^2+4H_iH_i)}]/2=2\mu+\tilde{\lambda}_2,$$

$$\lambda_3=\lambda_4=\lambda_5=\lambda_6=2\mu;$$

$$t_{ip} = \begin{bmatrix} \dfrac{H_1 + \tilde\lambda_1/3}{\sqrt{H_sH_s + \tilde\lambda_1^2/3}} & \dfrac{H_1 + \tilde\lambda_2/3}{\sqrt{H_sH_s + \tilde\lambda_2^2/3}} & \dfrac{H_3 - H_2}{\sqrt{3H_sH_s}} & 0 & 0 & 0 \\ \dfrac{H_2 + \tilde\lambda_1/3}{\sqrt{H_sH_s + \tilde\lambda_1^2/3}} & \dfrac{H_2 + \tilde\lambda_2/3}{\sqrt{H_sH_s + \tilde\lambda_2^2/3}} & \dfrac{H_1 - H_3}{\sqrt{3H_sH_s}} & 0 & 0 & 0 \\ \dfrac{H_3 + \tilde\lambda_1/3}{\sqrt{H_sH_s + \tilde\lambda_1^2/3}} & \dfrac{H_3 + \tilde\lambda_2/3}{\sqrt{H_sH_s + \tilde\lambda_2^2/3}} & \dfrac{H_2 - H_1}{\sqrt{3H_sH_s}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

$$c_1 = \frac{H_3 - H_1}{H_2 - H_1}, \qquad c_3 = \frac{H_2 + \tilde\lambda_1/3}{H_1 + \tilde\lambda_1/3}.$$

The matrix (50) is positive definite if the following necessary and sufficient conditions are satisfied:

$$2\mu(3\lambda + 2\mu) > 3(H_1^2 + H_2^2 + H_3^2), \qquad \mu > 0.$$

It is of interest to find, in addition to materials of the form (36), (41), other anisotropic materials that admit purely transverse waves, for example, when the wave displacement vector contains the septor S_{mkl} [see (25) and (26)]. For this, it is necessary to solve system (10) and study the properties of the septor. This is a subject for further study.

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